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Representation of solutions of boundary value problems for nonlinear evolution equations by special series with recurrently calculated coefficients

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Abstract. One of the analytical methods of presenting solutions of nonlinear partial differential equations is the method of special series in powers of specially selected functions called basic functions. The coefficients of such series are found successively as solutions of linear differential equations. The basic functions can also contain arbitrary functions. By using such functional arbitrariness allows in some cases, to prove the global convergence of the corresponding constructed series, and also allows us to prove the solvability of the boundary value problem for the Korteweg-de Vries equation. In the paper for a nonlinear wave equation a theorem on the possibility of satisfying a given boundary condition using an arbitrary function contained in the basic function is proved.

1. Introduction

Method of special series [1, 2, 3] is a method of representation of solutions of nonlinear partial differential equations in the form of series by the powers of one or several functions chosen in a special way, which allow the series coefficients to be calculated recurrently without applying any truncation procedures. The method of special series, in contrast to methods of the Galerkin type [4, 5], makes it possible to find a solution with a controlled accuracy, since the approaches used lead to a chain of finite-dimensional systems of ordinary differential equations that turn out to be linear even for nonlinear solvable equations, which allows us to obtain new results: it is possible to prove global convergence constructed series in unbounded domains of the [2, 6, 7], where the application of numerical methods has fundamental difficulties.

In some cases it is possible to exactly satisfy zero boundary conditions by the choice of the basic functions (for example, for description of nonlinear vibrating string with fixed end points [1], or membrane with fixed edges [8]). In other cases it is possible to satisfy a predetermined boundary condition by using the functional arbitrariness contained in the basic functions [9, 10, 11]. Application series with recurrently calculated coefficients allowed to obtain space-periodic solutions of the Navier-Stockes system [12] and to prove existence of analytical solutions for various problems [13, 14, 15].

In some cases it was possible to construct the basis functions that take into account presence of a known exact solutions, as well as the specific character of non-linear equations. For example, these series were built and studied in [16, 17]. Sometimes constructed series turn into finite sums and then obtained the exact solution [18, 19].

2. The menhod of special series with recurrently calculated coefficients

Let us consider one of constructions of special series for solving Cauchy problem for nonlinear partial differential equations of the form

$$u_t = F\left(t, u, \frac{\partial u}{\partial x}, \cdots, \frac{\partial^m u}{\partial x^m}\right), \quad u(0,x) = u^0(x), \tag{1}$$

where F is a polynomial of the unknown function $u(t,x)$ and its derivatives with respect to the space variable. Follow [20, 21], we describe the scheme of constructing special series for basic functions with functional arbitrariness.

The solution is represented by the series

$$u(t,x) = \sum_{n=0}^{\infty} u_n(t) P^n(t,x) \tag{2}$$

by the powers of basic function $P(t,x)$ satisfying the overdetermined system

$$P_x = A(t,P), \quad P_t = B(t,P) \tag{3}$$

with functions $A(t,P)$ and $B(t,P)$ being analytic respect to P and such that $A(t,0) \equiv 0$ and $B(t,0) \equiv 0$. It was shown that if the initial conditions are written in the form

$$u^0(x) = \sum_{n=0}^{\infty} u_n^0 P^n(0,x), \tag{4}$$

then substituting series (2) into equation (1), collecting similar terms, and taking into account relations (3), we obtain the sequence of first-order ordinary differential equations for the coefficients $u_n(t)$

$$u_n^{'} = F_n(t, u_n, \ldots, u_0), \qquad u_n(0) = u_n^0, \quad n = 1, 2, \ldots$$

where the right-hand sides F_n include only u_j with $j \leq n$ and the coefficients u_n may be linearly contained only.

3. Proof of existence of a solution of an initial boundary value problem for nonlinear wave equation in the form of special series

Consider the use of special series with basic functions with functional arbitrariness to construction of solutions of of nonlinear wave equation. Solution for qeneralized Korteweg–de Vries equations

$$\frac{\partial u}{\partial t} + \gamma \frac{\partial^{2r+1} u}{\partial x^{2r+1}} + F\left(t, u, \ldots, \frac{\partial^{2r} u}{\partial x^{2r}}\right) = 0, \quad \gamma = \text{const} \tag{5}$$

with initial condition $u(0,x) = u^0(x)$ in the form of (4) and boundary condition at $x = 0$

$$u(0,t) = h(t), \quad h(t) \in C^1[0,\infty), \quad h(t) - u_0 \neq 0, \quad t \geq 0. \tag{6}$$

was constructed as special series (2) in the powers of the basic function

$$\begin{aligned} &P(x,t) = (\exp(bx) + f(t))^{-1}, \quad f(t) = f_0 + \int_0^t \varphi(\tau) d\tau, \\ &\varphi(t) \in C[0,\infty), \quad f_0 > 0, \quad b = \text{const}. \end{aligned} \tag{7}$$

Basic function (7) satisfies system (3) for

$$A(t,P){=}{-}bP{+}bf(t)P^2, \quad B(t,P){=}{-}f(t)'P^2$$

with an arbitrary function $f(t)$.

For $b<0$ the following theorem was proved [9].

Theorem 1. *Let the following conditions be fulfilled:*

1. $|u_n^0| \le Mn^{-m}, \quad 0<M\le M_0, \quad n\ge 1, \quad m=2r+4, \quad u_0>0;$
2. $0<\varphi(t)\le M_1, \quad t\ge 0.$

Then there exists a unique function $\varphi(t)\in C[0,T_0]$, $M_0, M_1, T_0 = \mathrm{const}$ *such that problem* (5), (4) *and* (6) *has a solution for* $-\infty<x\le 0$ *and* $0\le t\le T_0$.

3.1. Nonlinear wave equations

Let consider initial-boundary problem for nonlinear wave equation

$$u_{tt}=u_{xx}+G(t,u), \tag{8}$$

with initial data

$$u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x). \tag{9}$$

and boundary conditions

$$u(0,t)=h(t), \quad u(x,t)\to 0 \quad \text{by} \quad x\to+\infty. \tag{10}$$

Here $G(t,u)$ is a polynomial by the unknown function $u(x,t)$ with the coefficients which are continuous functions of t. Let initial conditions (9) are represented in the form of convergent series

$$u_\nu(x)=\sum_{n=0}^{\infty}u_{n,\nu}P^n(0,x), \quad u_{n,\nu}=\mathrm{const}, \quad \nu=0,1. \tag{11}$$

by the powers of function

$$\begin{aligned} &P(x,t)=(\exp(bx)+f(t))^{-1}, \quad f(t)=f_0+f_1t+\int_0^t\int_0^t\varphi(\tau)d\tau d\tau, \\ &\varphi(t)\in C[0,\infty), \quad f_0>0, \quad f_1>0, \quad b>0. \end{aligned} \tag{12}$$

In [21] was proved that a solution of problem (8), (9), (11) also may be construct in the form of the series (2), (12).

The coefficients $u_n(t)$ after substitution of series (2), (12) into (8) and equating the expressions of the same powers $P(x,t)$, will be found from a sequence of linear second order differential equations

$$u_n''-b^2n^2u_n=L_n(t,f)+R_n(t,f), \quad n\ge 0, \tag{13}$$

where $L_n(t,f)$ is the following expression

$$L_n(t,f){=}f''u_{n-1}{+}2(n{-}1)f'u_{n-1}'{-}b^2f(n{-}1)nu_{n-1}{-}b^2(n{-}1)^2u_{n-1}$$

$$+[b^2f^2{-}(f')^2](n{-}2)(n{-}1)u_{n-2},$$

related with the linear terms in equation (8), and $R_n(t)$ is an expression, related with nonlinear function $G(t,u)$. By this the coefficients $u_k(t)$ of series (2), (12) are included into expressions $L_n(t,f)$ and $R_n(t,f)$ with $k\le n-1$, that allows to find the coefficients of the series successively.

Initial conditions for the equation (13) are determined by constant $u_{n,\nu}$, $\nu = 1, 2$ in initial conditions (9)

$$u_n(0) = u_{n,0}, \quad u_n'(0) = u_{n,1} + (n-1)f'(0), \quad n \geq 1. \tag{14}$$

Solutions of the sequence of linear second order differential equations (13), (14) have the form

$$u_n(t,f){=}A_n \exp(bht){+}B_n \exp(-bnt){+}\frac{1}{bn}\int\limits_0^t \operatorname{sh}(bn(\tau{-}t))(L_n(\tau,f){+}R_n(\tau,f))d\tau, \quad n \geq 1,$$

where constants A_n and B_n are determined from initial conditions (14)

$$A_n + B_n = u_{n,0}, \quad A_n bn - B_n bn = u_{n,1}.$$

For the coefficients of series (2), (12) the following estimations are obtained.

$$|u_n(t,f)| \leq \frac{M\exp(2bnt)}{n^4}, \quad n \geq 1, \tag{15}$$

$$|u_n'(t,f)| \leq \frac{Mb\exp(2bnt)}{n^3}, \quad n \geq 1, \tag{16}$$

$$|u_n''(t,f)| \leq \frac{4Mb^2\exp(2bnt)}{n^2}, \quad n \geq 1. \tag{17}$$

The following theorem is valid.

Theorem 2. *Let the following conditions be fulfilled:*

1. $|u_{n,0}| \leq \dfrac{M}{6n^4}, \quad |u_{n,1}| \leq \frac{Mb}{6n^3}, \quad M > 0, \quad n \geq 1;$
2. $f(t) \in C^2[0,\infty)$, $0 < q_1 \leq f(t) \leq q$, $|f'(t)| \leq q$, $|f''(t)| \leq q, \quad q_1, q > 0, \quad t \geq 0.$

Then there are exist constants $M_0 > 0$ and $q_0 > 0$, that for $M \leq M_0$ and $q \leq q_0$ series (2), (12) *converges to a solution of Cauchy problem* (8), (9), (11) *for all $x \geq 0$ and $0 \leq t \leq T$, where $T = \frac{\ln(1+q_0)}{2b}$.*

In [21] to prove this theorem the estimations (15)–(17) are used and it was noted that, the question on satisfying a given boundary condition for nonlinear wave equations by choosing function $f(t)$, included into basic function (12), is currently still open.

Let us proceed to the study of the existence of the solution of the initial-boundary value problem (8), (9), (10) in the form of series (2), (12). We assume that $h(t) \in C^2[0,\infty)$,$u_0 = 0$ and $h(t) \neq 0$. Let for $x = 0$ and $t = 0$ the compatibility condition must be fulfilled. To satisfy boundary condition (10) for $t \geq 0$ it is necessary to show that there exists a function $f(t)$ such that the equality

$$h(t) = \sum_{n\geq 1} \frac{u_n(t,f(t))}{(1+f(t))^n} \tag{18}$$

is valid. To show the solvability of the initial-boundary value problem (8)–(10) we perform auxiliary constructions [9]. Multiplying equation (18) by $[1{+}f_0{+}f_1 t{+}\int\limits_0^t\int\limits_0^t \varphi(\tau)d\tau]/h(t)$ and differentiating the obtained equality twice with respect to t, we obtain

$$\begin{aligned}\varphi(t) &= a(t) + \sum_{n\geq 2} \frac{1}{(1+f)^{n-1}}[b_0(t)u_n - b_1(t)u_n^{'} + b_2(t)u_n^{''}] \\ &+ \sum_{n\geq 2} \frac{1}{(1+f)^n}[b_1(t)(n-1)u_n f^{'} - 2b_2(t)(n-1)u_n^{'} f^{'} - b_2(t)(n-1)u_n f^{''}] \\ &+ \sum_{n\geq 2} \frac{b_2(t)n(n-1)(f^{'})^2}{(1+f)^{n+1}},\text{`}\end{aligned} \tag{19}$$

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where

$$a(t) = \Big[\frac{u_1^{'}}{h} - \frac{u_1 h^{'} + u_1 h^{''}}{h^2} + \frac{2u_1 h^{'} - u_1^{'} h^{'}}{h^3}\Big], \quad b_0 = \frac{2h^{'} - h^{''}}{h^2},$$

$$b_1(t) = \frac{h^{'}}{h^2}, \quad b_2(t) = \frac{1}{h}, \quad u_1(t) = A_1 \exp(bt) + B_1 \exp(-bt).$$

Let us consider the space of continuous on $[0, T]$ functions with the norm $||\varphi||_T = \max\limits_{0 \le t \le T} |\varphi(t)|$.

The set

$$\Omega_T = \{\varphi : \quad \varphi \in C[0, T], \quad |\varphi(t)| \le D, \quad (D = \text{const}), \quad 0 \le t \le T\}$$

is closed, bounded, and complete.

Denote by $A(\varphi)$ the operator which is equal to the right-hand side of (19) and prove that this operator is a contraction on the space Ω_T.

The following lemmas are valid.

Lemma 1. *For natural n there exists T_2 $(0 < T_2 \le T_1)$ such that the estimates*

$$\max_{0 \le t \le T_2} \left| \frac{1}{(1+f(t))^n} - \frac{1}{(1+g(t))^n} \right| \le n T_2 ||\varphi - \psi||_{T_2}$$

are valid.

Lemma 2. *Under the conditions of Theorem 2 with $M_1 = 1 + q_1$ there exists T_3 $(0 < T_3 \le T_2)$ that the following estimates of the coefficients u_n of the series are valid:*

$$\max_{0 \le t \le T_3} |u_n(t, f) - u_n(t, g)| \le \frac{M_1^n}{n^3} ||\varphi - \psi||_{T_3}$$

Lemma 3. *Let conditions of Lemma 2 be fulfilled and $h(t) \in C^2[0, T_3]$, $0 < H_1 \le |h(t)| \le H_2$, $|h'(0)| \le H_2$, $|h''(0)| \le H_2$, $H_1 > 0$, $H_2 > 0$. Then there exist numbers $T_4 > 0$ and $M_0 > 0$ such that if $0 < M \le M_0$, then the operator A maps the set Ω_{T_4} into itself.*

Lemma 4. *Let conditions of Lemma 2 be fulfilled and $h(t) \in C^2[0, T_3]$, $0 < H_1 \le |h(t)| \le H_2$, $|h'(0)| \le H_2$, $|h''(0)| \le H_2$, $H_1 > 0$, $H_2 > 0$. Then there exist numbers $T_4 > 0$ and $M_0 > 0$ such that if $0 < M \le M_0$, then the operator A maps the set Ω_{T_4} into itself.*

Lemma 5. *Let the conditions of Lemma 4 be fulfilled, then there exist numbers $\widehat{M} > 0$ and $T_5 > 0$ such that if $0 < M \le \widehat{M}$, then the operator A is a contraction on the space Ω_{T_5}, i.e.,*

$$||A(\varphi) - A(\psi)||_{T_5} \le L||\varphi - \psi||_{T_5}, \quad L < 1, \quad \varphi, \psi \in \Omega_{T_5}.$$

Proving this lemmas, we also use a particular case of Lemma 2 in [1] which may be written in the form

$$\sum_{m_1 + m_2 = N} \frac{1}{m_1^q m_2^q} \le \frac{4^q \pi^2}{3N^q},$$

where m_1, $m_2 \ge 1$, and $q \ge 2$.

Taking into account the lemmas 5 and 6 we can prove the following theorem.

Theorem 3. *Let the conditions of Lemma* 6 *be fulfilled. Then there exists a unique function* $\varphi \in \Omega_{T_5}$ *such that initial-boundary problem* (8)*,* (9)*,* (11)*,* (10) *has a solution for* $x \geq 0$ *and* $0 \leq t \leq T_5$*.*

Proof. By Lemmas 5 and 6 and the principle of contraction mapping, equation (19) has a unique solution $\varphi \in \Omega_{T_5}$, i.e., the boundary condition (18) for $x = 0$ is satisfied on the time interval $0 \leq t \leq T_5$.

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